



Working Paper 13-16
Statistics and Econometrics Series (15)
May 2013

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MULTIPERIOD PORTFOLIO SELECTION WITH TRANSACTION AND MARKET-IMPACT COSTS

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Abstract

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Keywords: Portfolio optimization; Multiperiod utility; No-trade region

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Acknowledgements: Mei and Nogales are supported by the Spanish Government through project MTM2010-16519

Multiperiod Portfolio Selection with Transaction and Market Impact Costs

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We carry out an analytical investigation on the optimal portfolio policy for a multiperiod mean-variance investor facing multiple risky assets. We consider the case with proportional, market impact, and quadratic transaction costs. For proportional transaction costs, we find that a buy-and-hold policy is optimal: if the starting portfolio is outside a parallelogram-shaped no-trade region, then trade to the boundary of the no-trade region at the first period, and hold this portfolio thereafter. For market impact costs, we show that the optimal portfolio policy at each period is to trade to the boundary of a state-dependent movement region. Moreover, we find that the movement region shrinks along the investment horizon, and as a result the investor trades throughout the entire investment horizon. Finally, we show numerically that the utility loss associated with ignoring transaction costs or investing myopically may be large.

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1. Introduction

Mossin (1968), Samuelson (1969), and Merton (1969, 1970) show how an investor should optimally choose her portfolio in a dynamic environment in the absence of transaction costs. In practice, however, implementing a dynamic portfolio policy requires one to rebalance the portfolio weights frequently, and this may result in high transaction costs. To address this issue, researchers have tried to characterize the optimal portfolio policies in the presence of transaction costs. The case with a single-risky asset and proportional transaction costs is well understood. In a multiperiod setting, Constantinides (1979) shows that the optimal trading policy is characterized by a no-trade interval, such that if the risky-asset portfolio weight is inside this interval, then it is optimal not to trade, and if the portfolio weight is outside, then it is optimal to trade to the boundary of this interval. Later, Constantinides (1986) and Davis and Norman (1990) extend this result to a continuous-time setting with a single risky asset.

The case with multiple risky assets is much harder to characterize, and the existent literature is sparse. Akian et al. (1996) consider a multiple risky-asset version of the continuous time model by Davis and Norman (1990), and using *simulations* they suggest that the optimal portfolio policy is characterized by a multi-dimensional no-trade region. Leland (2000) develops a relatively simple *numerical* procedure to compute the no-trade region based on the existence results of Akian et al. (1996). The only paper that provides an analytical characterization of the no-trade region for the case with multiple risky assets is the one by Liu (2004), who shows that under the assumption that asset returns are *uncorrelated*, the optimal portfolio policy is characterized by a separate no-trade interval for each risky asset.

But none of the aforementioned papers characterizes analytically the no-trade region for the general case with multiple risky assets with correlated returns and proportional transaction costs. The reason for this is that the analysis in these papers relies on modelling the *asset return* distribution, and as a result they must take portfolio growth into account, which renders the problem untractable analytically. Recently, Garleanu and Pedersen (2012) consider a setting that relies on modelling *price changes*, and thus they are able to give closed-form expressions for the optimal dynamic portfolio policies in the presence of *quadratic* transaction cost. Arguably, modelling price changes is not very different from modelling stock returns, at least for daily or higher trading frequencies, yet the former approach renders the problem tractable.

We make three contributions. Our first contribution is to use the multiperiod framework proposed by Garleanu and Pedersen (2012) to characterize the optimal portfolio policy for the general case with multiple risky assets and proportional transaction costs. Specifically, we show that there exists a no-trade region, shaped as a parallelogram, such that if the starting portfolio is inside the no-trade region, then it is optimal not to trade at any period. If, on the other hand, the starting portfolio is outside the no-trade region, then it is optimal to trade to the boundary of the no-trade region in the first period, and not to trade thereafter. Furthermore, we study how the no-trade region depends on the level of proportional transaction costs, the correlation in asset returns, the discount factor, the investment horizon, and the risk-aversion parameter.

Our second contribution is to study analytically the optimal portfolio policy in the presence of market impact costs, which arise when the investor makes large trades that distort market prices. Traditionally, researchers have assumed that the market *price* impact is linear on the amount traded (see Kyle (1985)), and thus that market impact costs are quadratic. Under this assumption, Garleanu and Pedersen (2012) derive closed-form expressions for the optimal portfolio policy within their multiperiod setting. However, Torre and Ferrari (1997), Grinold and Kahn (2000), and Almgren et al. (2005) show that the square root function is more appropriate for modelling market price impact, thus suggesting market impact costs grow at a rate slower than quadratic. Our contribution is to extend the analysis by Garleanu and Pedersen (2012) to the

case where market impact costs follow a power function with an exponent between 1 and 2. For this case, we show that there exists an analytical *movement region* for every time period, such that the optimal policy at each period is to trade to the boundary of the corresponding movement region. Thus we find that, unlike with proportional transaction costs, it is optimal for the investor to trade at every period when she faces market impact costs.

Finally, our third contribution is to show numerically that the utility losses associated with ignoring transaction costs and behaving as a myopic investor are large.

Our work is related to Dybvig (2005), who considers a single-period setting with mean-variance utility and proportional transaction costs. For the case with multiple risky assets, he shows analytically that the optimal portfolio policy is characterized by a no-trade region shaped as a parallelogram, but the manuscript does not provide a formal rigorous proof. Like Dybvig (2005), we consider proportional transaction costs and mean-variance utility, but we extend the results to a multi-period setting, and provide a complete rigorous proof.

This manuscript is organized as follows. Section 2 describes the multiperiod framework under general transaction costs. Section 3 studies the case with proportional transaction costs, Section 4 the case with market impact costs, and Section 5 the case with quadratic transaction costs. Section 6 characterizes numerically the utility loss associated with ignoring transaction costs, and with behaving myopically. Section 7 concludes.

2. General Framework

Our framework is closely related to that proposed by Garleanu and Pedersen (2012); herein G&P. Like G&P, we consider a multiperiod setting, where the investor tries to maximize her discounted mean-variance utility net of transaction costs by choosing the number of shares to hold of each of the N risky assets. There are three main differences between our model and the model by G&P. First, we consider a more general class of transaction costs that includes not only quadratic transaction costs, but also proportional and market impact costs. Second, we assume price changes are independent and identically distributed with mean μ and covariance matrix Σ , while G&P consider the more general case in which price changes are predictable. Third, we consider both the finite and infinite horizon cases, whereas G&P focus on the infinite horizon case.

The investor's objective is

$$\max_{\{x_{t+i}\}_{i=0}^{T-1}} \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} (x_{t+i}^T \mu - \frac{\gamma}{2} x_{t+i}^T \Sigma x_{t+i}) - (1-\rho)^i \kappa \|x_{t+i} - x_{t+i-1}\|_p^p \right], \quad (1)$$

where $x_{t+i} \in \mathbb{R}^N$ contains the number of shares of each of the N risky assets held in period $t+i$, T is the investment horizon, ρ is the discount factor, and γ is the risk-aversion parameter. The term $\kappa \|x_{t+i} - x_{t+i-1}\|_p^p$ is the transaction cost for the $(t+i)$ th period, where $\kappa_j \in \mathbb{R}$, and $\|s\|_p$ is the p-norm of vector s ; that is, $\|s\|_p^p = \sum_{i=1}^N |s_i|^p$. We consider the cases with proportional transaction costs ($p = 1$), quadratic transaction costs ($p = 2$), and market impact costs ($p \in (1, 2)$).

To simplify the exposition, we focus on the case where the transaction costs associated with trading all N risky assets are symmetric. It is straightforward, however, to extend our results to the case where the transaction costs associated with different assets are asymmetric; that is, the case where the transaction costs associated with trading the j th asset at the $(t+i)$ th period is: $\kappa_j |x_{t+i,j} - x_{t+i-1,j}|^{p_j}$.

Our analysis relies on the following assumption.

ASSUMPTION 1. *Price changes are independently and identically distributed with mean μ and covariance matrix Σ .*

3. Proportional Transaction Costs

In this section we consider the case where transaction costs are proportional to the amount traded (that is, $p = 1$). These so-called proportional transaction costs are appropriate to model the cost associated with trades that are small, and thus the transaction cost originates from the bid ask spread and other commissions charges by brokers. For exposition purposes, we first study the single-period case, and show that for this case the optimal portfolio policy is characterized by a no-trade region shaped as a parallelogram.¹ We then study the general multiperiod case, and again show that there is a no-trade region shaped as a parallelogram. Moreover, if the starting portfolio is inside the no-trade region, then it is optimal not to trade at any period. If, on the other hand, the starting portfolio is outside the no-trade region, then it is optimal to trade to the boundary of the no-trade region in the first period, and not to trade thereafter. Furthermore, we study how the no-trade region depends on the level of proportional transaction costs, the correlation in asset returns, the discount factor, the investment horizon, and the risk-aversion parameter.

3.1. The Single-Period Case

For the single-period case, the investor's decision is

$$\max_x (1 - \rho)(x^T \mu - \frac{\gamma}{2} x^T \Sigma x) - \kappa \|x - x_0\|_1, \quad (2)$$

where x_0 is the starting portfolio.

¹ Our analysis provides a complete rigorous proof for the analysis in Dybvig (2005).

Unlike for the case with quadratic transaction costs, it is not possible to obtain closed-form expressions for the optimal portfolio policy for the case with proportional transaction costs. The following proposition, however, demonstrates that the optimal trading policy is characterized by a no-trade region shaped as a parallelogram.

PROPOSITION 1. *Let Assumption 1 hold, then:*

1. *The investor's decision problem (2) can be equivalently rewritten as:*

$$\min_x (x - x_0)^T \Sigma (x - x_0), \quad (3)$$

$$s.t. \quad \|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma}, \quad (4)$$

where $x^* = \Sigma^{-1}\mu/\gamma$ is the optimal portfolio in the absence of transaction costs (the Markowitz or target portfolio), and $\|s\|_\infty$ is the infinity norm of vector s ; that is, $\|s\|_\infty = \max_i \{|s_i|\}$.

2. *Constraint (4) defines a no-trade region shaped as a parallelogram centered at the target portfolio x^* , such that if the starting portfolio x_0 is inside this region, then it is optimal not to trade, and if the starting portfolio is outside this no-trade region, then it is optimal to trade to the point in the boundary of the no-trade region that minimizes the objective function in (3).*

It is easy to see that the size of no-trade region defined by Equation (4) decreases with the risk aversion parameter γ . Intuitively, the more risk averse the investor, the larger her incentives to trade and diversify her portfolio. Also, it is clear that the size of the no-trade region increases with the proportional transaction parameter κ . This makes sense intuitively because the larger the transaction cost parameter, the less attractive to the investor is to trade in order to move closer to the target portfolio. Moreover, the following proposition shows that there exists a finite transaction cost parameter κ^* such that if the transaction cost parameter $\kappa > \kappa^*$, then it is optimal not to trade.

PROPOSITION 2. *The no-trade region is unbounded when $\kappa \geq \kappa^*$, where $\kappa^* = \|\phi\|_\infty$ and ϕ the vector of Lagrange multipliers associated with the constraint in the following optimization problem:*

$$\max_x (1 - \rho)(x^T \mu - \frac{\gamma}{2} x^T \Sigma x), \quad (5)$$

$$s.t. \quad x - x_0 = 0. \quad (6)$$

Figure 1 depicts the parallelogram-shaped no-trade region together with the level sets for the objective function in problem eq3–eq4. The optimal portfolio policy is to trade to the intersection between the no-trade region and the tangent level set.

3.2. The Multiperiod Case

In this section, we show that similar to the single-period case, the optimal portfolio policy for the multiperiod case is also characterized by a no-trade region shaped as a parallelogram and centered around the target portfolio. If the starting portfolio at the first period is inside this no-trade region, then it is optimal not to trade at any period. If, on the other hand, the starting portfolio at the first period is outside this no-trade region, then it is optimal to trade to the boundary of the no-trade region in the first period, and not to trade thereafter. Summarizing, we find that in the multiperiod case with proportional transaction costs, it is optimal to either trade only at the first period, or not at all.

The investor's decision for this case can be written as:

$$\max_{\{x_{t+i}\}_{i=0}^{T-1}} \left\{ \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} \left(x_{t+i}^T \mu - \frac{\gamma}{2} x_{t+i}^T \Sigma x_{t+i} \right) - (1-\rho)^i \kappa \|x_{t+i} - x_{t+i-1}\|_1 \right] \right\}. \quad (7)$$

The following theorem demonstrates that the optimal trading policy is characterized by a no-trade region shaped as a parallelogram.

THEOREM 1. *Let Assumption 1 hold, then:*

1. *It is optimal not to trade at any period other than the first period; that is,*

$$x_t = x_{t+1} = \dots = x_{t+T-1}. \quad (8)$$

2. *The investor's optimal portfolio for the first period x_t (and thus for all subsequent periods) is the solution to the following constrained optimization problem:*

$$\min_{x_t} (x_t - x_{t-1})^T \Sigma (x_t - x_{t-1}), \quad (9)$$

$$s.t. \quad \|\Sigma(x_t - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T}. \quad (10)$$

where x_{t-1} is the starting portfolio, and $x^* = \Sigma^{-1}\mu/\gamma$ is the optimal portfolio in the absence of transaction costs (the Markowitz or target portfolio).

3. *Constraint (10) defines a no-trade region shaped as a parallelogram centered at the target portfolio x^* , such that if the starting portfolio x_{t-1} is inside this region, then it is optimal not to trade at any period, and if the starting portfolio is outside this no-trade region, then it is optimal to trade at the first period to the point in the boundary of the no-trade region that minimizes the objective function in (9), and not to trade thereafter.*

The following corollary establishes how the size of the no-trade region for the multiperiod case depends on the problem parameters.

COROLLARY 1. *The no-trade region for a multiperiod investor defined in (10) has the following properties:*

- *The no-trade region expands as proportional transaction parameter κ increases.*
- *The no-trade region expands as discount factor parameter ρ increases.*
- *The no-trade region shrinks as investment horizon T increases.*
- *The no-trade region shrinks as risk-aversion parameter γ increases.*

Similar to the single-period case, we observe that the size of the no-trade region grows with the transaction cost parameter κ . This is intuitive as the larger the transaction costs, the less willing the investor is to trade in order to diversify. This is illustrated in Figure 2, which depicts the no-trade regions for different values of the transaction cost parameter κ for a case with two stocks. In addition, it is easy to show following the same argument used to prove Proposition 2 that there is a κ^* such that the no-trade region is unbounded for $\kappa \geq \kappa^*$.

The size of the no-trade region increases with the discount factor ρ . Again, this makes sense intuitively because the larger the discount factor, the less important the utility for future periods and thus the smaller the incentive to trade today. This is illustrated in Figure 3a. The size of the no-trade region decreases with the investment horizon T . To see this intuitively, note that we have shown that the optimal policy is to trade at the first period and hold this position thereafter. Then, a multiperiod investor with shorter investment horizon cares will be more concerned about the transaction costs incurred at the first stage, compared with the investor who has a longer investment horizon. Finally, when $T \rightarrow \infty$, the no-trade region shrinks to the parallelogram bounded by $\frac{\kappa\gamma}{(1-\rho)\gamma}$, which is much closer to the center x^* . Of course, when $T = 1$, the multiperiod problem reduces to the static case. This is illustrated in Figure 3b.

In addition, the no-trade region shrinks as the risk aversion parameter γ increases. Intuitively, as the investor becomes more risk averse, the optimal policy is to move closer to the safe position x^* , despite the transaction costs associated with this. This is illustrated in Figure 4a, which also shows that the target portfolio changes with the risk-aversion parameter, and therefore the no-trade regions are centered at different points for different risk-aversion parameters. The no-trade region also depends on the correlation between assets. Figure 4b shows the no-trade regions for different correlations. When two assets are positively correlated, the parallelogram leans to the left while with negative correlation it leans to the right. In the absence of correlations the no-trade region becomes a rectangle.

4. Market Impact Costs

In this section we consider market impact costs, which arise when the investor makes large trades that distort market prices. Traditionally, researchers have assumed that the market *price* impact is linear on the amount traded (see Kyle (1985)), and thus that market impact costs are quadratic. Under this assumption, Garleanu and Pedersen (2012) derive closed-form expressions for the optimal portfolio policy within their

multiperiod setting. However, Torre and Ferrari (1997), Grinold and Kahn (2000), Almgren et al. (2005), and Gatheral (2010) show that the square root function is more appropriate for modelling market price impact, thus suggesting market impact costs grow at a rate slower than quadratic. Therefore in this section we consider the case with $p \in (1, 2)$ in objective function (1).

4.1. The Single-Period Case

For the single-period case, the investor's decision is:

$$\max_x (1 - \rho)(x^T \mu - \frac{\gamma}{2} x^T \Sigma x) - \kappa \|x - x_0\|_p^p, \quad (11)$$

where $1 < p < 2$.

Problem (11) can be solved numerically, but unfortunately it is not possible to obtain closed-form expressions for the optimal portfolio policy. The following proposition, however, shows that the optimal portfolio policy is to trade to the boundary of a *movement region* that depends on the starting portfolio and contains the target or Markowitz portfolio.

PROPOSITION 3. *Let Assumption 1 hold, then if the starting portfolio x_0 is equal to the target or Markowitz portfolio x^* , the optimal policy is not to trade. Otherwise, it is optimal to trade to the boundary of the following movement region:*

$$\frac{\|\Sigma(x - x^*)\|_q}{p\|x - x_0\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}, \quad (12)$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Comparing Theorems 1 and 3 we identify three main differences between the cases with proportional and market impact costs. First, for the case with market impact costs it is always optimal to trade (except in the trivial case where the starting portfolio coincides with the target or Markowitz portfolio), whereas for the case with proportional transaction costs it may be optimal not to trade if the starting portfolio is inside the no-trade region. Hence, we term the region defined by Equation (12) as a *movement region*, rather than a no-trade region. Second, the movement region depends on the starting portfolio x_0 , whereas the no-trade region is independent of it. Third, the movement region contains the target or Markowitz portfolio, but it is not centered around it, whereas the no-trade region is centered around the Markowitz portfolio.

In addition, note that the size of the movement region increases with the transaction cost parameter κ , and decreases with the risk-aversion parameter. The intuition for these results is similar to that for the case with proportional transaction costs. Finally, Figure 5 depicts the movement region and the optimal portfolio policy for a particular two-asset example. The figure shows that the movement region is a convex region containing the Markowitz portfolio.

4.2. The Multiperiod Case

The investor's decision for this case can be written as:

$$\max_{\{x_{t+i}\}_{i=0}^{T-1}} \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} \left(x_{t+i}^T \mu - \frac{\gamma}{2} x_{t+i}^T \Sigma x_{t+i} \right) - (1-\rho)^i \kappa \|x_{t+i} - x_{t+i-1}\|_p^p \right]. \quad (13)$$

As in the single-period case, it is not possible to provide closed-form expressions for the optimal portfolio policy, but the following theorem illustrates the analytical properties of the optimal portfolio policy.

THEOREM 2. *Let Assumption 1 hold, then:*

1. *if the starting portfolio x_{t-1} is equal to the target or Markowitz portfolio x^* , the optimal policy is not to trade at any period,*
2. *otherwise it is optimal to trade at every period. Moreover, at the i th period it is optimal to trade to the boundary of the following movement region:*

$$\frac{\left\| \sum_{j=i}^{T-1} (1-\rho)^{j-i} \Sigma (x_{t+j} - x^*) \right\|_q}{p \|x_{t+i} - x_{t+i-1}\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}, \quad (14)$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2 shows that for the multiperiod case with market impact costs it is optimal to trade at every period (except in the trivial case where the starting portfolio coincides with the Markowitz portfolio). Moreover, at every period it is optimal to trade to the boundary of a movement region that depends in the starting portfolio as well as the portfolio for every subsequent period. Therefore, the optimal portfolio for every period depends not only on the portfolio for the previous period, but also on the portfolio for every subsequent period. Finally, note that the size of the movement region for period i , assuming the portfolios for the rest of the periods are fixed, increases with the transaction cost parameter κ and decreases with the discount factor ρ and the risk-aversion parameter γ .

The following proposition shows that the movement region for period i contains the movement region for every subsequent period.

PROPOSITION 4. *Let Assumption 1 hold, then:*

1. *the movement region for the i th period contains the movement region for every subsequent period,*
2. *every movement region contains the Markowitz portfolio,*
3. *the movement region converges to the Markowitz portfolio in the limit when the investment horizon goes to infinity.*

Figure 6 shows the optimal portfolio policy and the movement regions for an example with an investment horizon $T = 3$. The figure confirms that the movement region for each period contains the movement region

for the subsequent periods. Moreover, the Markowitz portfolio x^* is contained in every movement region. For each stage, any trade is to the boundary of the movement region and the movement is towards the Markowitz strategy x^* . Note also that the figure shows that it is optimal for the investor to buy the second asset in the first period, and then sell it in the second period. This may appear suboptimal from the point of view of market impact costs, but it turns out to be optimal when the investor considers the trade off between multiperiod mean-variance utility and market impact costs.

5. Quadratic Transaction Costs

We now consider the case with quadratic transaction costs. Following G&P, we consider the following investor's decision:

$$\max_{\{x_{t+i}\}_{i=0}^{T-1}} \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} (x_{t+i}^T \mu - \frac{\gamma}{2} x_{t+i}^T \Sigma x_{t+i}) - (1-\rho)^i \|\Lambda^{1/2} (x_{t+i} - x_{t+i-1})\|_2^2 \right], \quad (15)$$

where Λ is a symmetric positive-definite matrix measuring the level of transaction costs.² Like G&P, we focus in the case where Λ is proportional to the covariance matrix Σ ; that is, $\Lambda = \lambda \Sigma / 2$. This framework differs from that considered by G&P in two respects: (i) G&P assume price changes are predictable, whereas we assume price changes are iid, and (ii) G&P consider an infinite horizon, whereas we allow for a finite investment horizon. Nevertheless, it is easy to adapt the results in G&P to provide an explicit characterization of the optimal portfolio policy.

THEOREM 3. *Let Assumption 1 hold and let $\Lambda = \lambda \Sigma / 2$, then:*

1. *The optimal portfolios $x_t, x_{t+1}, \dots, x_{t+T-1}$ satisfy the following linear equations:*

$$x_{t+i} = \alpha_1 x^* + \alpha_2 x_{t+i-1} + \alpha_3 x_{t+i+1}, \quad \text{for } i = 0, 1, \dots, T-2 \quad (16)$$

$$x_{t+i} = \beta_1 x^* + \beta_2 x_{t+i-1}, \quad \text{for } i = T-1. \quad (17)$$

where $\alpha_1 = \frac{(1-\rho)\gamma}{(1-\rho)\gamma + (2-\rho)\lambda}$, $\alpha_2 = \frac{\lambda}{(1-\rho)\gamma + (2-\rho)\lambda}$, $\alpha_3 = \frac{(1-\rho)\lambda}{(1-\rho)\gamma + (2-\rho)\lambda}$, with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, and $\beta_1 = \frac{(1-\rho)\gamma}{(1-\rho)\gamma + \lambda}$, $\beta_2 = \frac{\lambda}{(1-\rho)\gamma + \lambda}$, with $\beta_1 + \beta_2 = 1$.

2. *The optimal portfolio converges to the Markowitz portfolio as the investment horizon T goes to infinity.*
3. *The optimal portfolios for periods $t, t+1, \dots, t+T-1$ lay on a straight line.*

Theorem 3 shows that the optimal portfolio for each stage is a linear combination of the Markowitz strategy (the target portfolio), the previous period portfolio and the next period portfolio.

Figure 7 provides a comparison of the optimal portfolio policy for the case with quadratic transaction costs, with those for the cases with proportional and market impact costs, for a multiperiod investor with

² Note that the investor's decision (15) is a bit more general than our framework given in (1) because of the matrix Λ . For the case where $\Lambda = \kappa I$, where I is the identity matrix, we recover the framework in (1).

$T = 4$. The figure confirms that, for the case with quadratic transaction costs, the optimal portfolio policy is to trade at every period along a straight line that converges to the Markowitz portfolio. It can also be appreciated that the investor trades more aggressively at the first periods compared to the final periods. For the case with proportional transaction costs, it is optimal to trade to the boundary of the no-trade region shaped as a parallelogram in the first period and not to trade thereafter. Finally, for the case with market impact costs, the investor trades at every period to the boundary of the corresponding movement region. The resulting trajectory is not a straight line, moreover, as in the example discussed in Section 4, it is optimal for the investor to buy the second asset in the first period and then sell it in the second period; that is, the optimal portfolio policy is inefficient in terms of market impact costs, but it is optimal in terms of the tradeoff between market impact costs and discounted utility.

6. Numerical Analysis

We study numerically the utility loss associated with ignoring transaction costs and investing myopically, as well as how these utility losses depend on the transaction cost parameter, the risk-aversion parameter, the price change correlation, the investment horizon, and the number of assets. To simplify the discussion, we focus on the case with proportional transaction costs.

To do this, we consider three different portfolio policies. First, we consider the *target* portfolio policy, which consists of trading to the target or Markowitz portfolio in the first period and not trading thereafter. This is the optimal portfolio policy for an investor in the absence of transaction costs. Second, the *static* portfolio policy, which consists of trading in the first period to the solution to problem (2), and not trading thereafter. This is the optimal portfolio policy for a myopic investor who takes into account transaction costs. Third, we consider the *multiperiod* portfolio policy, which is the optimal portfolio policy for a multiperiod investor who takes into account transaction costs; that is, the solution to problem (7).

We evaluate the utility of each of the three portfolio policies in the multiperiod framework with proportional transaction costs given by the objective function equation (7).

6.1. Base case

We consider a case with proportional transaction costs of 50 basis points, risk-aversion parameter $\gamma = 10^{-5}$, which corresponds to a relative risk aversion of 1 for an investor managing $M = 10^5$ dollars³, annual discount factor $\rho = 5\%$, and an investment horizon of $T = 22$ days (one month). We consider four risky assets ($N = 4$), with starting price of 1 dollar, asset price change correlations of 0.2, and we assume the

³ Garleanu and Pedersen (2012) consider an absolute risk aversion $\gamma = 10^{-9}$, which corresponds to an investor managing $M = 10^9$ dollars

starting portfolio is equally-weighted across the four risky assets with a total number of $M = 10^5$ shares. We randomly draw the annual average price changes from a uniform distribution with support $[0.1, 0.25]$, and the annual price change volatilities from a uniform distribution with support $[0.1, 0.4]$.

For our base case, we observe that the utility loss associated with investing myopically (that is, the difference between the utility of the *multiperiod* portfolio policy and the *static* portfolio policy) is 54.33%. The utility loss associated with ignoring transaction costs altogether (that is, the difference between the utility of the *multiperiod* portfolio policy and the *target* portfolio policy) is 58.39%. Hence we find that the loss associated with either ignoring transaction costs or behaving myopically can be large.

6.2. Comparative statics

We study numerically how the utility loss associated with ignoring transaction costs and investing myopically depend on the transaction cost parameter, the risk-aversion parameter, the price change correlation, the investment horizon, and the number of assets.

Figure 8 depicts the utility loss associated with investing myopically and ignoring transaction costs for values of the proportional transaction costs parameter κ ranging from 10 basis point to 110 basis points. We find that behaving myopically results in utility losses of around 80% for transaction costs of 10 basis points—because the static portfolio policy trades too little compared with the multiperiod portfolio policy—, and the utility losses decrease monotonically with the level of transaction costs—because in the limit as the transaction costs grow large, both the static and multiperiod policies result in little or no trading. The utility loss associated with ignoring transaction costs is obviously zero for the case without transaction costs and increases monotonically with transaction costs. Moreover, for large transaction costs parameters, the utility loss associated with ignoring transaction costs grows linearly with κ and can be very large.

Regarding the risk-aversion parameter, our numerical results show that the utility losses associated with investing myopically and ignoring transaction costs do not depend on the risk-aversion parameter.

Figure 9 depicts the utility loss associated with investing myopically and ignoring transaction costs for values of price change correlation ranging from -0.3 to 0.4 . The utility losses associated with behaving myopically range from 40% to 95% and decrease monotonically with correlation. The reason for this is that for high correlation the benefits from diversification are smaller, and thus the utility difference between the static and multiperiod portfolio policies are smaller. The utility loss associated with ignoring transaction costs remains relatively constant around 60% for correlations smaller than 0.2 and increases for higher levels of correlation. This makes sense intuitively again because the diversification benefits associated with trading are larger for smaller correlation, and thus the performance of the target portfolio deteriorates for large correlations.

Figure 10 depicts the utility loss associated with investing myopically and ignoring transaction costs for investment horizons ranging from $T = 5$ (one week) to $T = 100$ (20 weeks). Not surprisingly, the utility loss associated with behaving myopically grows with the investment horizon. Also, the utility loss associated with ignoring transaction costs is very large for the single-period case, and decreases monotonically with the investment horizon. The reason for this is that the size of the no-trade region for the multiperiod portfolio policy decreases monotonically with the investment horizon, and thus the target and multiperiod policies become similar. As the investment horizon increases, the utility for multiperiod model approaches to the Markowitz strategy. This is intuitive, by adopting Markowitz strategy, a multiperiod investor loses money only at the first stage and makes profit for the rest of the infinite horizon, hence the transaction costs that she may incur is negligible compared with the profit she may earn.

Figure 11 depicts the utility loss associated with investing myopically and ignoring transaction costs for number of assets ranging from $N = 4$ to $N = 100$. The utility losses associated with ignoring transaction costs and behaving myopically increase with the number of risky assets, being the latter larger.

7. Conclusions

We consider the optimal portfolio policy for a multiperiod mean-variance investor facing multiple risky assets subject to proportional, market impact, or quadratic transaction costs.

We demonstrate analytically that, in the presence of proportional transaction costs, the optimal strategy for the multiperiod investor is to trade in the first period to the boundary of a no-trade region shaped as a parallelogram, and not to trade thereafter. For the case with market impact costs, the optimal portfolio policy is to trade to the boundary of a state-dependent movement region. In addition, the movement region converges to the Markowitz portfolio as the investment horizon grows large.

We contribute to the literature by characterizing the no-trade region for a multiperiod investor facing proportional transaction costs. In addition, we study the analytical properties of the optimal trading strategy for the model with market impact costs. Finally, we show numerically that the utility losses associated with ignoring transaction costs or investing myopically may be large.

Appendix. Figures

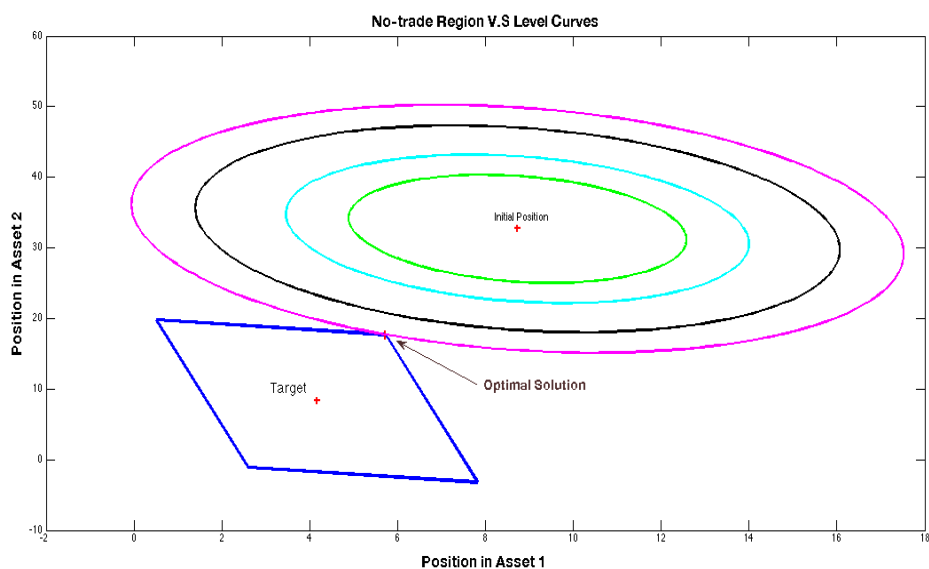


Figure 1 No-trade region and level sets for objective function 3.

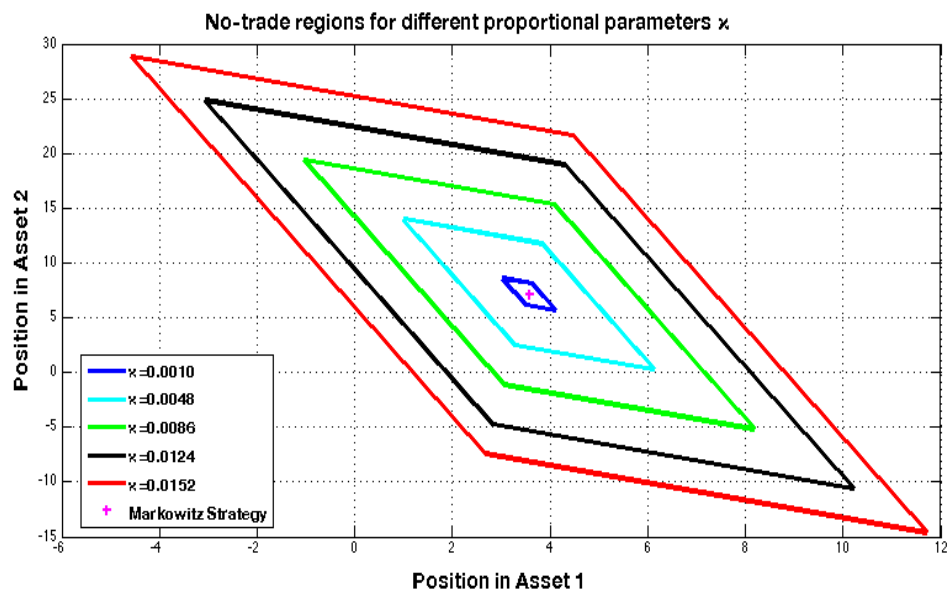
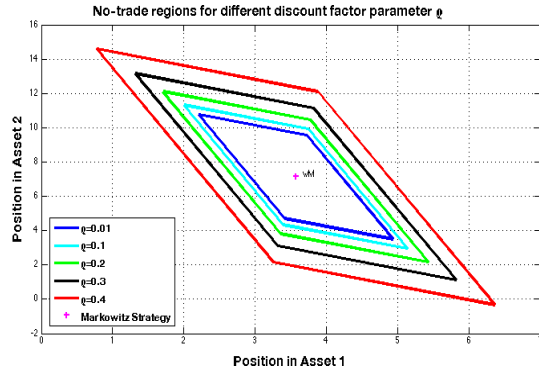
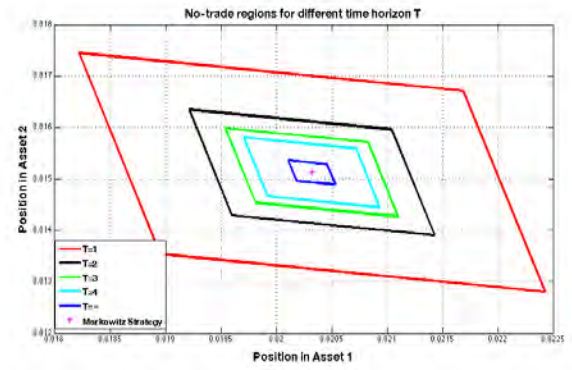
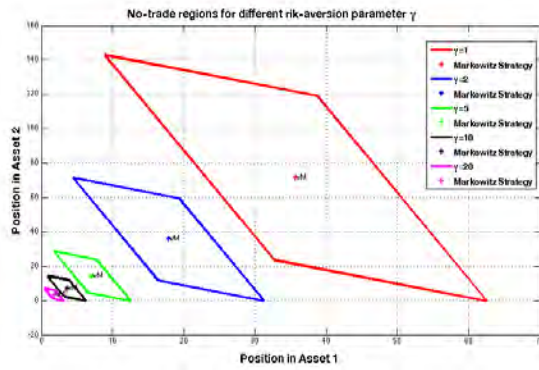
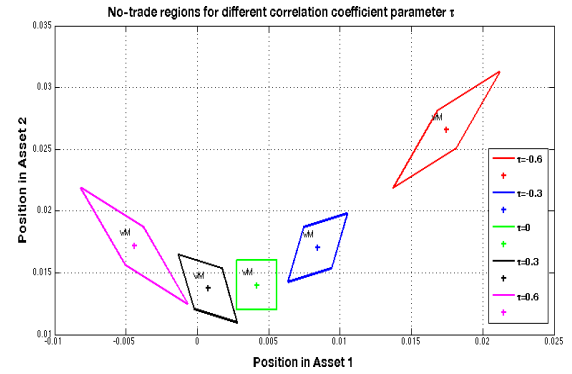
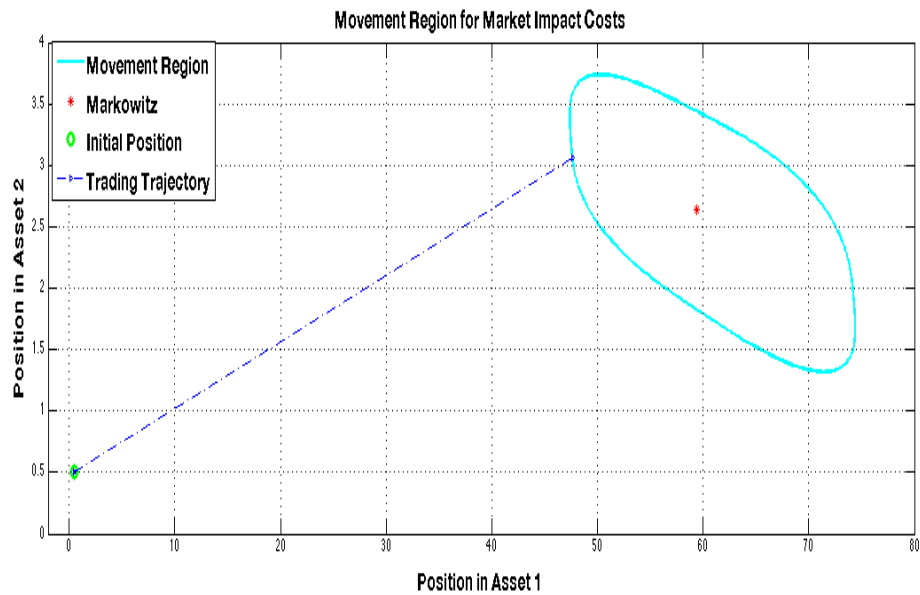


Figure 2 No-trade regions for different values of κ .

(a) No-trade regions depending on ρ (b) No-trade regions depending on T **Figure 3** No-trade regions for different discount factors and investment horizons(a) No-trade regions depending on γ 

(b) No-trade regions depending on correlation

Figure 4 No-trade regions for different risk aversion and correlations**Figure 5** Movement Region for Market Impact Costs for a Myopic Investor

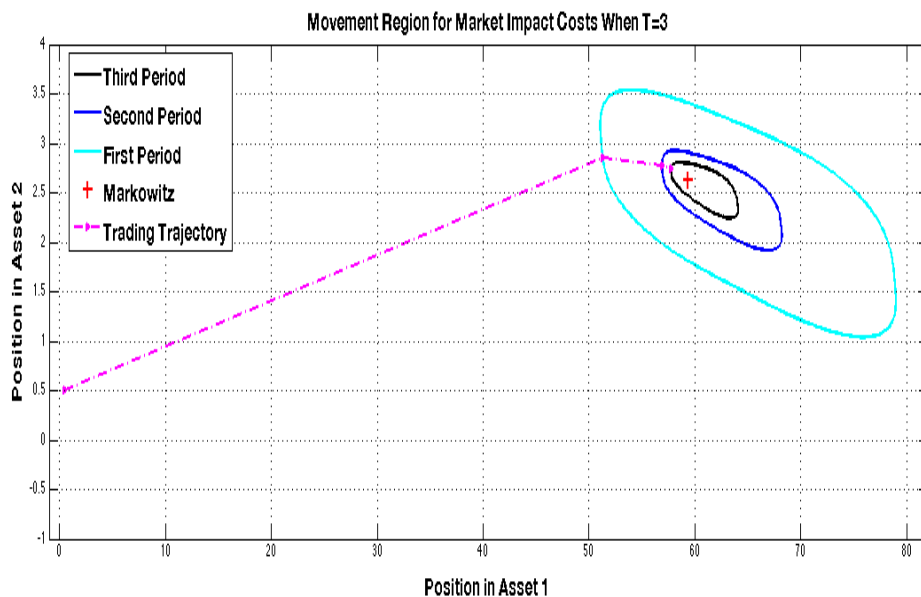


Figure 6 Movement Regions for Market Impact Costs

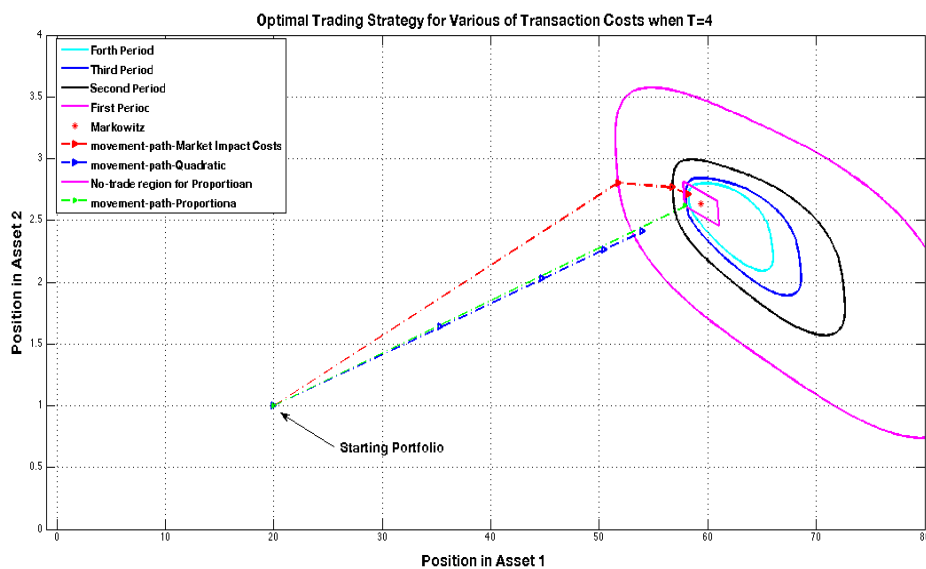


Figure 7 Trading trajectory for different transaction costs.
This figure depicts the optimal trading trajectory for the cases with proportional transaction costs, market impact costs, and quadratic transaction costs.

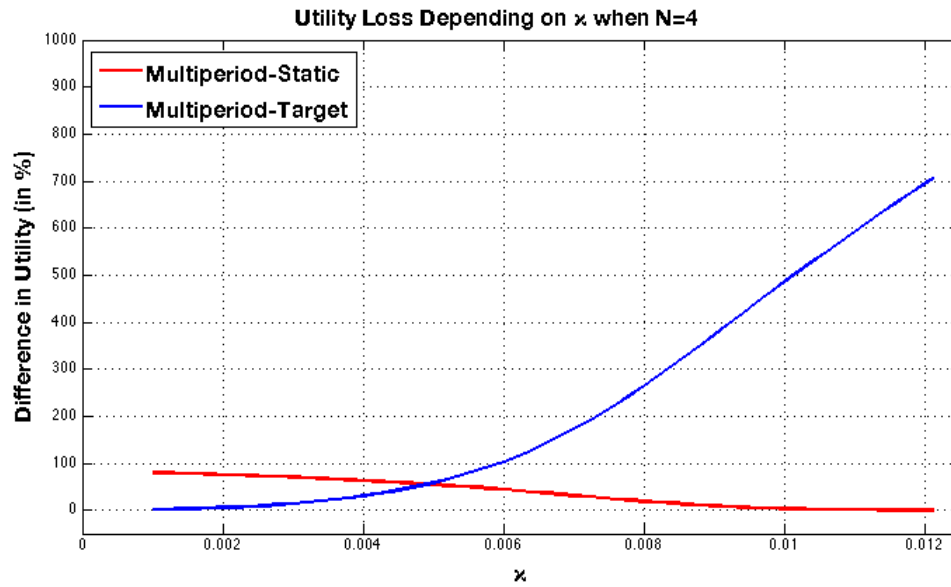


Figure 8 Utility Losses Depending on transaction cost parameter κ .

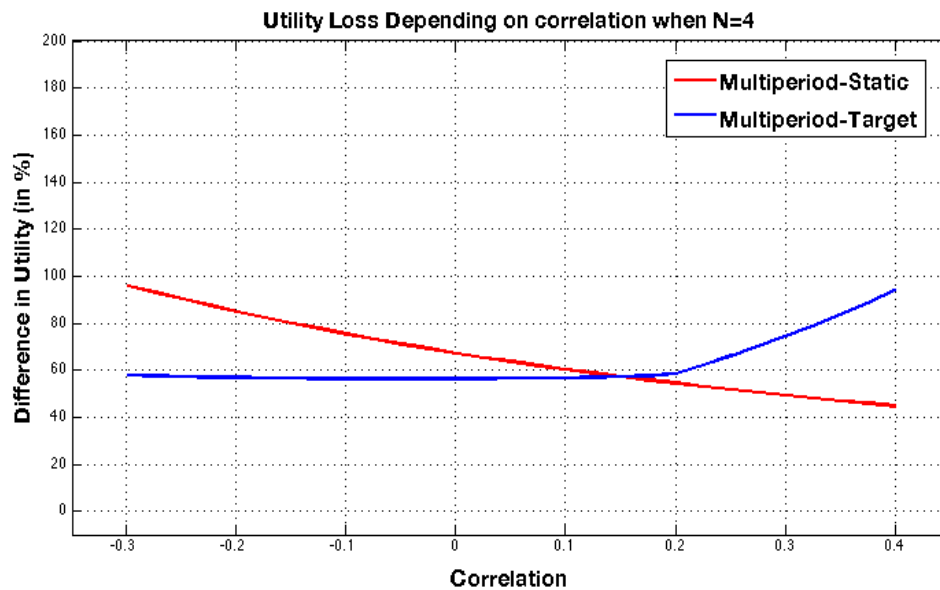


Figure 9 Utility Losses Depending on Correlation.

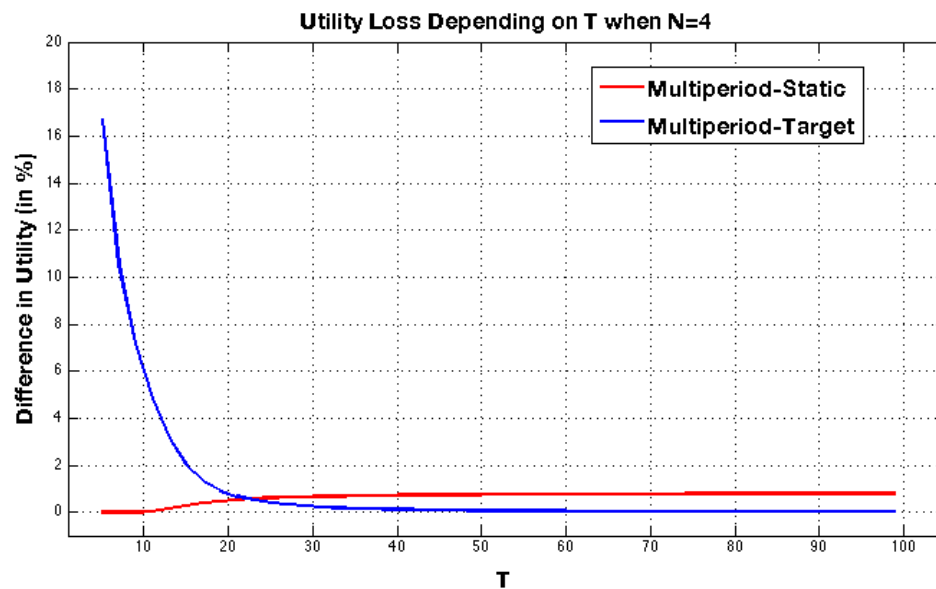


Figure 10 Utility Losses Depending on Investment Horizon T .

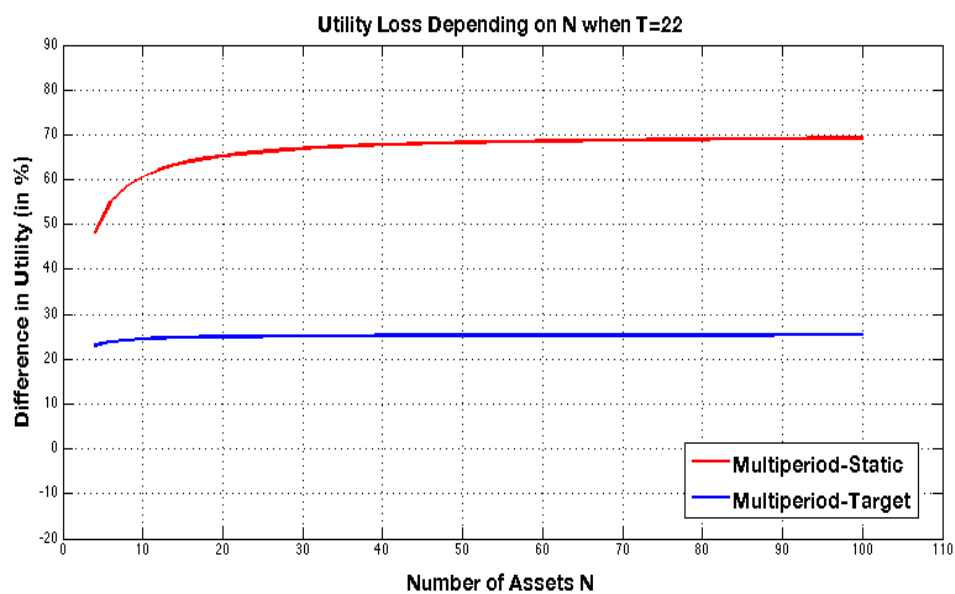


Figure 11 Utility Losses Depending on Number of Assets N .

Appendix. Proofs

Proof of Proposition 1

Let a subgradient of $\|x - x_0\|_1$ be denoted as s , and the corresponding subdifferential Ω , that is:

$$s \in \Omega = \{u \mid u^T(x - x_0) = \kappa\|x - x_0\|_1, \|u\|_\infty \leq \kappa\} \quad (18)$$

With this definition, we can write $\|x - x_0\|_1 = \max_{\|s\|_\infty \leq \kappa} s^T(x - x_0)$, and hence the function (3) can be expressed as:

$$\begin{aligned} & \max_x (1 - \rho)(x^T \mu - \frac{\gamma}{2} x^T \Sigma x) - \kappa \|x - x_0\|_1 \\ &= \max_x \min_{\|s\|_\infty \leq \kappa} (1 - \rho)(x^T \mu - \frac{\gamma}{2} x^T \Sigma x) - s^T(x - x_0) \\ &= \min_{\|s\|_\infty \leq \kappa} \max_x (1 - \rho)(x^T \mu - \frac{\gamma}{2} x^T \Sigma x) - s^T(x - x_0). \end{aligned} \quad (19)$$

The optimal value of $\max_x (1 - \rho)(x^T \mu - \frac{\gamma}{2} x^T \Sigma x) - s^T(x - x_0)$ will satisfy the following optimality condition:

$$0 = (1 - \rho)(\mu - \gamma \Sigma x) - s, \quad (20)$$

and hence:

$$x = \frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s), \quad (21)$$

which can be written in the form $x = \frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s)$ for some $s \in \Omega$.

Writing now the optimal expression for x in (18), the following expression is attained:

$$\begin{aligned} & \min_{\|s\|_\infty \leq \kappa} \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s) \right]^T \mu - \frac{\gamma}{2} \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s) \right]^T \Sigma \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s) \right] \\ & \quad - s^T \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s) - x_0 \right] \\ & \equiv \min_{\|s\|_\infty \leq \kappa} \frac{1}{2\gamma} (\mu - \frac{1}{1 - \rho} s)^T \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s) + s^T x_0. \end{aligned} \quad (22)$$

Because the optimal solution satisfies $0 \in \mu - \gamma \Sigma x - \frac{1}{1 - \rho} s$, that means the subgradient s must be $s = (1 - \rho)(\mu - \gamma \Sigma x)$ as long it satisfies $\|s\|_\infty \leq \kappa$.

Then writing the expression $s = (1 - \rho)(\mu - \gamma \Sigma x) = (1 - \rho)[\gamma \Sigma(x^* - x)]$ back into the objective function (21), we can conclude that problem (21) is equivalent to the following problem:

$$\begin{aligned} & \min_x \quad \frac{\gamma}{2} x^T \Sigma x - \gamma x^T \Sigma x_0. \\ & s.t. \quad \|(1 - \rho)\gamma \Sigma(x - x^*)\|_\infty \leq \kappa. \end{aligned} \quad (23)$$

Rearranging terms, we can prove problem (2) is equivalent to:

$$\max_x \quad \frac{\gamma}{2} x^T \Sigma x + x_0^T \mu - \gamma x^T \Sigma x_0. \quad (24)$$

$$s.t. \quad \|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (25)$$

Because the term $\frac{\gamma}{2}x_0^T\Sigma x_0$ is constant:

$$\begin{aligned}
& \frac{\gamma}{2}x^T\Sigma x - \gamma x^T\Sigma x_0 \\
= & \frac{\gamma}{2}x^T\Sigma x - \gamma x^T\Sigma x_0 + \frac{\gamma}{2}x_0^T\Sigma x_{t-1} \\
= & \frac{\gamma}{2}(x - x_0)^T\Sigma(x - x_0) \\
= & (x - x_0)^T\Sigma(x - x_0).
\end{aligned} \tag{26}$$

Finally, adding the constraint (25), we conclude problem (3)-(4) is equivalent to:

$$\min_x (x - x_0)^T\Sigma(x - x_0), \tag{27}$$

$$s.t. \quad \|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma}. \tag{28}$$

To show that constraint (4) defines a no-trade region, note that when the initial position x_0 satisfies constraint (4), then $x = x_0$ minimizes the objective function and is feasible with respect to the constraint. On the other hand, when x_0 is not in the region defined by the constraint, the optimal x must be the point in the boundary of the feasible region that minimizes the objective. Finally, to see that the no-trade region defined by constraint (4) is a parallelogram centered around x^* , note that constraint $\|\Sigma(x_t - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma}$ is equivalent to $-\frac{\gamma}{(1-\rho)\kappa}e \leq \Sigma(x_t - x^*) \leq \frac{\gamma}{(1-\rho)\kappa}e$. \square

Proof of Proposition 2

Because the l_1 norm is a exact penalty function, then the optimal solution of the problem (5)-(6) is also the optimal solution of

$$\max_x (1-\rho)(x^T\mu - \frac{\gamma}{2}x^T\Sigma x) + \kappa\|x - x_0\|_1, \tag{29}$$

for any $\kappa \geq \kappa^*$, where $\kappa^* = \|\phi\|_\infty$. \square

Proof of Theorem 1

Part 1. Let the subgradient of $\|x_{t+i} - x_{t+i-1}\|_1$ be s_{i+1} , and the subdifferential Ω_{i+1} :

$$s_{i+1} \in \Omega_{i+1} = \{u_{i+1} \mid u_{i+1}^T(x_{t+i} - x_{t+i-1}) = \kappa\|x_{t+i} - x_{t+i-1}\|_1, \|u_{i+1}\|_\infty \leq \kappa\}, \tag{30}$$

for for $i = 0, 1, \dots, T-1$. Since we can rewrite $\|x_{t+i} - x_{t+i-1}\|_1 = \max_{\|s_{i+1}\|_\infty \leq \kappa} s_{i+1}^T(x_{t+i} - x_{t+i-1})$, objective function (7) can be expressed as:

$$\begin{aligned}
& \max_{\{x_{t+i}\}_{i=0}^{T-1}} \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} \left(x_{t+i}^T\mu - \frac{\gamma}{2}x_{t+i}^T\Sigma x_{t+i} \right) - (1-\rho)^i \kappa \|x_{t+i} - x_{t+i-1}\|_1 \right] \\
= & \max_{\{x_{t+i}\}_{i=0}^{T-1}} \min_{\|s_{i+1}\|_\infty \leq \kappa} \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} \left(x_{t+i}^T\mu - \frac{\gamma}{2}x_{t+i}^T\Sigma x_{t+i} \right) - (1-\rho)^i s_{i+1}^T(x_{t+i} - x_{t+i-1}) \right] \\
= & \min_{\|s_{i+1}\|_\infty \leq \kappa} \max_{\{x_{t+i}\}_{i=0}^{T-1}} \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} \left(x_{t+i}^T\mu - \frac{\gamma}{2}x_{t+i}^T\Sigma x_{t+i} \right) - (1-\rho)^i s_{i+1}^T(x_{t+i} - x_{t+i-1}) \right].
\end{aligned} \tag{31}$$

The optimality conditions, respect to x_{t+i} , of the inside subproblem

$$f_{obj-inside} = \max_{\{x_{t+i}\}_{i=0}^{T-1}} \sum_{i=0}^{T-1} \left[(1-\rho)^{i+1} \left(x_{t+i}^T\mu - \frac{\gamma}{2}x_{t+i}^T\Sigma x_{t+i} \right) - (1-\rho)^i s_{i+1}^T(x_{t+i} - x_{t+i-1}) \right], \tag{32}$$

satisfy:

$$0 = (1 - \rho)(\mu - \gamma \Sigma x_{t+i}) - s_{i+1} + (1 - \rho)s_{i+2}, \quad (33)$$

where $s_{i+1} \in \Omega_{i+1}$. Condition (33) ensures us to write x_{t+i} in the form of

$$x_{t+i} = \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{i+2}) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_{i+1}, \quad (34)$$

for some $s_{i+1} \in \Omega_{i+1}$. Denote the optimal solution by x_{t+i}^* , then there exists some s_{i+1}^* such that:

$$x_{t+i}^* = \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{i+2}^*) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_{i+1}^*, \quad \forall \quad i \quad (35)$$

To prove that $x_t = x_{t+1} = \dots = x_{t+T-1}$ satisfies the first order condition, we need to determine the values of s_{t+i}^* satisfying $\|s_{t+i}^*\|_\infty \leq \kappa$ and $x_{t+i}^* = x_{t+j}^*$ for all $i \neq j$. It further indicates s_{i+1}^* and s_{j+1}^* satisfies:

$$\frac{1}{\gamma} \Sigma^{-1}(\mu + s_{i+2}^*) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_{i+1}^* = \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{j+2}^*) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_{j+1}^*, \quad \forall \quad i, j \quad (36)$$

Hence the value of s_{j+1} is $s_{j+1}^* = \frac{1 - (1 - \rho)^{T-j+1}}{\rho} s_T^*$ with $\|s_{j+1}^*\|_\infty \leq \kappa$ for $j = 0, 1, \dots, T - 2$. Since

$$\frac{1 - (1 - \rho)^{T-j+1}}{\rho} > \frac{1 - (1 - \rho)}{\rho} = 1, \quad (37)$$

we can deduce $s_T^* \leq \kappa$.

So if we define $s_{i+1}^* = \frac{1 - (1 - \rho)^{T-j+1}}{\rho} s_T^*$, for $i = 0, 1, \dots, T - 2$ with $s_T^* = (1 - \rho)(\mu - \gamma \Sigma x_{t+T-1}^*)$ satisfying $\|s_T^*\|_\infty \leq \kappa$, we can conclude that $x_t = x_{t+1} = \dots = x_{t+T-1}$ satisfies the optimality conditions.

Part 2. Simplify objective function (7) into the following way based on $x_t = x_{t+1} = \dots = x_{t+T-1}$:

$$\begin{aligned} & \max_{\{x_{t+i}\}_{i=0}^{T-1}} \left\{ \sum_{i=0}^{T-1} \left[(1 - \rho)^{i+1} \left(x_{t+i}^T \mu - \frac{\gamma}{2} x_{t+i}^T \Sigma x_{t+i} \right) - (1 - \rho)^i \kappa \|x_{t+i} - x_{t+i-1}\|_1 \right] \right\} \\ &= \max_{x_t} \left\{ \sum_{i=0}^{T-1} \left[(1 - \rho)^{i+1} \left(x_t^T \mu - \frac{\gamma}{2} x_t^T \Sigma x_t \right) \right] - \kappa \|x_t - x_{t-1}\|_1 \right\} \\ &= \max_{x_t} \left(\sum_{i=0}^{T-1} (1 - \rho)^{i+1} \right) \left(x_t^T \mu - \frac{\gamma}{2} x_t^T \Sigma x_t \right) - \kappa \|x_t - x_{t-1}\|_1 \\ &= \max_{x_t} \frac{(1 - \rho) - (1 - \rho)^{T+1}}{\rho} \left(x_t^T \mu - \frac{\gamma}{2} x_t^T \Sigma x_t \right) - \kappa \|x_t - x_{t-1}\|_1. \end{aligned} \quad (38)$$

Now define subgradient of $\|x_t - x_{t-1}\|_1$ as s and the subdifferential Ω :

$$s \in \Omega = \{u \mid u^T(x_t - x_{t-1}) = \kappa \|x_t - x_{t-1}\|_1, \|u\|_\infty \leq \kappa\}. \quad (39)$$

Based on the same calculation as we did in the proof for Theorem 1 (substitute $(1 - \rho)$ in Theorem 1 with $\frac{(1 - \rho) - (1 - \rho)^{T+1}}{\rho}$ in this multiperiod case) we conclude objective function (38) is equivalent to:

$$\min_{x_t} (x_t - x_{t-1})^T \Sigma (x_t - x_{t-1}), \quad (40)$$

$$s.t. \quad \|\Sigma(x_t - x^*)\|_\infty \leq \frac{\kappa}{\gamma} \frac{\rho}{1 - \rho - (1 - \rho)^{T+1}}. \quad (41)$$

Rewrite the right-hand side of constraint (41):

$$\|\Sigma(x_t - x^*)\|_\infty \leq \frac{\kappa\rho}{(1-\rho)\gamma} \frac{1}{1-(1-\rho)^T}. \quad (42)$$

Then we can attain the final equivalence (9)-(10):

$$\min_{x_t} (x_t - x_{t-1})^T \Sigma (x_t - x_{t-1}), \quad (43)$$

$$s.t. \quad \|\Sigma(x_t - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1-(1-\rho)^T}. \quad (44)$$

Part 3. Apparently constraint (10) is a parallelogram centered at x^* since it is equivalent to

$$-\frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1-(1-\rho)^T} e \leq \Sigma(x_t - x^*) \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1-(1-\rho)^T} e.$$

To show that constraint (10) defines a no-trade region, note that when the starting portfolio x_{t-1} satisfies constraint (10), then $x_t = x_{t-1}$ minimizes the objective function (9) and is feasible with respect to the constraint. On the other hand, when x_{t-1} is not inside the region defined by (10), the optimal solution x_t must be the point on the boundary of the feasible region that minimizes the objective. By this means, constraint (10) defines a no-trade region. \square

Proof of Proposition 3

The optimality conditions for the objective function are:

$$(1-\rho)(\mu - \gamma \Sigma x) - \kappa p |x - x_0|^{p-1} \cdot \text{sign}(x - x_0) = 0, \quad (45)$$

where $|x - x_0|^{p-1}$ denotes the absolute value to the power of $p-1$ for each component:

$$|x - x_0|^{p-1} = (|x_{.,1} - x_{0,1}|^{p-1}, |x_{.,2} - x_{0,2}|^{p-1}, \dots, |x_{.,N} - x_{0,N}|^{p-1}),$$

and $\text{sign}(x - x_0)$ is a vector containing the sign of each component for $x - x_0$.

Rearranging

$$(1-\rho)\gamma \Sigma(x^* - x) = \kappa p |x - x_0|^{p-1} \cdot \text{sign}(x - x_0) \quad (46)$$

and we can conclude the point $x = x_0$ can not be the optimal solution when the initial position x_0 satisfies that $x_0 = x^*$. Otherwise, (46) implies that the optimal strategy x satisfies:

$$\|\Sigma(x - x^*)\|_q = \frac{\kappa}{(1-\rho)\gamma} p \| |x - x_0|^{p-1} \cdot \text{sign}(x - x_0) \|_q, \quad (47)$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Since $\| |x - x_0|^{p-1} \cdot \text{sign}(x - x_0) \|_q = \|x - x_0\|_p^{p-1}$, we can conclude that the optimal strategy satisfies

$$\frac{\|\Sigma(x - x^*)\|_q}{p \|x - x_0\|_p^{p-1}} = \frac{\kappa}{(1-\rho)\gamma}. \quad (48)$$

That is, when the initial position satisfies $x_{t-1} = x^*$, the optimal strategy is not to trade, otherwise the optimal strategy satisfies (48). \square

Proof of Theorem 2

The optimality conditions for the objective function (13) respect to x_{t+i} are:

$$\begin{aligned} \frac{\partial f_{obj}}{\partial x_{t+i}} &= (1-\rho)^{i+1}(\mu - \gamma \Sigma x_{t+i}) - (1-\rho)^i p \kappa |x_{t+i} - x_{t+i-1}|^{p-1} \cdot \text{sign}(x_{t+i} - x_{t+i-1}) \\ &\quad + (1-\rho)^{i+1} p \kappa |x_{t+i+1} - x_{t+i}|^{p-1} \cdot \text{sign}(x_{t+i+1} - x_{t+i}) = 0. \end{aligned} \quad (49)$$

For the last period, they reduce to

$$(1-\rho)(\mu - \gamma \Sigma x_{t+T-1}) - p \kappa |x_{t+T-1} - x_{t+T-2}|^{p-1} \cdot \text{sign}(x_{t+T-1} - x_{t+T-2}) = 0, \quad (50)$$

where the optimal x_{t+T-1} can not be equal to the previous position x_{t+T-2} unless $x_{t+T-2} = x^*$ in order to make the equation holds. Besides, when given $x_{t+T-1} = x_{t+T-2} = x^*$, it is convenient to show through (49) that $x_t = x_{t+1} = \dots = x_{t+T-1} = x^*$. Otherwise, we can take q -norm on both sides, and the last period optimal strategy satisfies

$$\frac{\|\Sigma(x_{t+T-1} - x^*)\|_q}{p \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (51)$$

Moreover, we can simplify each equation by adding terms recursively to obtain

$$p \kappa |x_{t+i} - x_{t+i-1}|^{p-1} \cdot \text{sign}(x_{t+i} - x_{t+i-1}) = \sum_{j=i}^{T-1} (1-\rho)^{j-i+1} \gamma \Sigma(x^* - x_{t+j}) \quad (52)$$

where again the optimal x_{t+i} can not be equal to x_{t+i-1} unless that $x_{t+i-1} = x^*$. Otherwise, we can take again the q -norm on both sides and the optimal strategy for each stage i satisfies,

$$\frac{\|\sum_{j=i}^{T-1} (1-\rho)^{j-i} \Sigma(x_{t+j} - x^*)\|_q}{p \|x_{t+i} - x_{t+i-1}\|_p^{p-1}} = \frac{\kappa}{(1-\rho)\gamma}. \quad (53)$$

If the optimal solution corresponds to any period satisfies $x_{t+i} = x_{t+i-1}$, it will leads to the contradiction that $x_{t-1} = 0$. Hence, we conclude the optimal strategy is always to move to (53) for each stage whenever the initial position is not x^* . \square

Proof of Proposition 4

Part 1. Define the function $g(x) = (1-\rho)\Sigma(x - x^*)$. Then, we know that for the last period $\|g(x_{t+T-1})\|_q \leq p \frac{\kappa}{\gamma} \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}$. Recursively, for the following last period $\|g(x_{t+T-2}) + (1-\rho)g(x_{t+T-1})\|_q \leq p \frac{\kappa}{\gamma} \|x_{t+T-2} - x_{t+T-3}\|_p^{p-1}$, and noting that $\|A + B\|_q \geq \|A\|_q - \|B\|_q$, we obtain

$$\begin{aligned} p \frac{\kappa}{\gamma} \|x_{t+T-2} - x_{t+T-3}\|_p^{p-1} &\geq \|g(x_{t+T-2}) + (1-\rho)g(x_{t+T-1})\|_q \\ &\geq \|g(x_{t+T-2})\|_q - (1-\rho)\|g(x_{t+T-1})\|_q \\ &\geq \|g(x_{t+T-2})\|_q - (1-\rho)p \frac{\kappa}{\gamma} \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}, \end{aligned}$$

where the last inequality holds because $\|g(x_{t+T-1})\|_q \leq p \frac{\kappa}{\gamma} \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}$.

Therefore,

$$\|g(x_{t+T-2})\|_q \leq p \frac{\kappa}{\gamma} \|x_{t+T-2} - x_{t+T-3}\|_p^{p-1} + (1-\rho)p \frac{\kappa}{\gamma} \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}$$

where we can deduce

$$\frac{\|g(x_{t+T-2})\|_q}{p\|x_{t+T-2} - x_{t+T-3}\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1-\rho) \frac{\kappa}{\gamma} \frac{\|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}}{\|x_{t+T-2} - x_{t+T-3}\|_p^{p-1}},$$

which is a wider area than the region defined for x_{t+T-1} : $\frac{\|g(x_{t+T-1})\|_q}{p\|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}} \leq \frac{\kappa}{\gamma}$.

Similarly, $\|g(x_{t+T-3}) + (1-\rho)g(x_{t+T-2}) + (1-\rho)^2g(x_{t+T-1})\|_q \leq \frac{\kappa}{\gamma} p\|x_{t+T-3} - x_{t+T-4}\|_p^{p-1}$, and then

$$\begin{aligned} \frac{\kappa}{\gamma} p\|x_{t+T-3} - x_{t+T-4}\|_p^{p-1} &\geq \|g(x_{t+T-3}) + (1-\rho)g(x_{t+T-2}) + (1-\rho)^2g(x_{t+T-1})\|_q \\ &\geq \|g(x_{t+T-3}) + (1-\rho)\|g(x_{t+T-2})\|_p - (1-\rho)^2\|g(x_{t+T-1})\|_q, \\ &\geq \|g(x_{t+T-3}) + (1-\rho)\|g(x_{t+T-2})\|_p - (1-\rho)^2 p \frac{\kappa}{\gamma} \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1} \end{aligned}$$

where the last inequality holds because $\|g(x_{t+T-1})\|_q \leq p \frac{\kappa}{\gamma} \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}$.

This implies that $\|g(x_{t+T-3}) + (1-\rho)\|g(x_{t+T-2})\|_p \leq \frac{\kappa}{\gamma} p\|x_{t+T-3} - x_{t+T-4}\|_p^{p-1} + (1-\rho)^2 p \frac{\kappa}{\gamma} \|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}$, Then we can show that,

$$\frac{\|g(x_{t+T-3}) + (1-\rho)\|g(x_{t+T-2})\|_p}{p\|x_{t+T-3} - x_{t+T-4}\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1-\rho)^2 \frac{\kappa}{\gamma} \frac{\|x_{t+T-1} - x_{t+T-2}\|_p^{p-1}}{\|x_{t+T-3} - x_{t+T-4}\|_p^{p-1}},$$

which is a region wider than the region defined by $\frac{\|g(x_{t+T-2}) + (1-\rho)\|g(x_{t+T-1})\|_p}{p\|x_{t+T-2} - x_{t+T-3}\|_p^{p-1}} \leq \frac{\kappa}{\gamma}$ for x_{t+T-2} .

Recursively, we can deduce the movement region corresponding to each period shrinks along t .

Part 2. Moreover, for each period i , the movement region relates with the trading strategies thereafter. The values for $x_{t+i} = x_{t+i+1} = \dots = x_{t+T-1} = x^*$ satisfies the inequality (14) leads to the fact that the movement region for stage i contains Markowitz strategy x^* .

Part 3. The optimality condition for the last period satisfies the following as has been shown in (50):

$$(1-\rho)(\mu - \gamma \Sigma x_{t+T-1}) - p\kappa|x_{t+T-1} - x_{t+T-2}|^{p-1} \cdot \text{sign}(x_{t+T-1} - x_{t+T-2}) = 0. \quad (54)$$

If there exists a limit for the policy x_{t+T-1} when $T \rightarrow \infty$, let ω be the vector such that $\lim_{T \rightarrow \infty} x_{t+T-1} = \omega$. Taking limit on both sides of (54):

$$(1-\rho)(\mu - \gamma \Sigma \omega) - p\kappa|\omega - \omega|^{p-1} \cdot \text{sign}(\omega - \omega) = 0, \quad (55)$$

since $\lim_{T \rightarrow \infty} x_{t+T-1} = \lim_{T \rightarrow \infty} x_{t+T-2} = \omega$. This indicates that:

$$(1-\rho)(\mu - \gamma \Sigma \omega) = 0$$

So $\omega = \frac{1}{\gamma} \Sigma^{-1} \mu = x^*$, which verifies the conclusion that the investor will eventually move to Markowitz strategy x^* . \square

Proof of Theorem 3

Part 1. The optimality conditions for (15) are

$$(1-\rho)(\mu - \gamma \Sigma x_{t+i}) - \frac{\lambda}{2}(2\Sigma x_{t+i} - 2\Sigma x_{t+i-1}) - \frac{\lambda(1-\rho)}{2}(2\Sigma x_{t+i} - 2\Sigma x_{t+i+1}) = 0, \quad (56)$$

which are equivalent to

$$[(1-\rho)\gamma \Sigma + \lambda \Sigma + (1-\rho)\lambda \Sigma] x_{t+i} = (1-\rho)\mu + \lambda \Sigma x_{t+i-1} + (1-\rho)\lambda \Sigma x_{t+i+1}. \quad (57)$$

We can then obtain the explicit expression for the optimal strategy x_{t+i} as following

$$x_{t+i} = \frac{1}{(1-\rho)\gamma + \lambda + (1-\rho)\lambda} [(1-\rho)\gamma x^* + \lambda x_{t+i-1} + (1-\rho)\lambda x_{t+i+1}], \quad (58)$$

where $i = 0, 1, \dots, T-2$. Then define $\alpha_1 = \frac{(1-\rho)\gamma}{(1-\rho)\gamma + (2-\rho)\lambda}$, $\alpha_2 = \frac{\lambda}{(1-\rho)\gamma + (2-\rho)\lambda}$, $\alpha_3 = \frac{(1-\rho)\lambda}{(1-\rho)\gamma + (2-\rho)\lambda}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ to conclude the result.

For $i = T-1$, the optimality condition for the last period is

$$(1-\rho)^T (\mu - \gamma \Sigma x_{t+T-1}) - \frac{\lambda}{2} (1-\rho)^{T-1} (2\Sigma x_{t+T-1} - 2\Sigma x_{t+T-2}) = 0, \quad (59)$$

Which is equivalent to

$$((1-\rho)\gamma + \lambda)\Sigma x_{t+T-1} = (1-\rho)\mu + \lambda\Sigma x_{t+T-2}. \quad (60)$$

We can then obtain the explicit solution for x_{t+T-1} as the following

$$x_{t+T-1} = \frac{(1-\rho)\gamma}{(1-\rho)\gamma + \lambda} x^* + \frac{\lambda}{(1-\rho)\gamma + \lambda} x_{t+T-2}. \quad (61)$$

Define $\beta_1 = \frac{(1-\rho)\gamma}{(1-\rho)\gamma + \lambda}$ and $\beta_2 = \frac{\lambda}{(1-\rho)\gamma + \lambda}$, we conclude the results.

Part 2. When $T \rightarrow \infty$, taking limit on both sides of (61) we can further verify that $\lim_{T \rightarrow \infty} x_{t+T-1} = x^*$. Then the investor will eventually trade to Markowitz strategy x^* .

Part 3. Moreover, the trajectory of the GP policy follows a straight line. First of all, equation (61) shows that x_{t+T-1} is a linear combination of x_{t+T-2} and Markowitz strategy x^* , it indicates that x_{t+T-1} , x_{t+T-2} and x^* are on the same straight line. If x_{t+T-3} is not on the same line, then x_{t+T-3} can not be expressed as linear combination of x_{t+T-1} , x_{t+T-2} and x^* . As a matter of fact, equation (58) shows that when $i = T-2$,

$$x_{t+T-2} = \alpha_1 x^* + \alpha_2 x_{t+T-3} + \alpha_3 x_{t+T-1}, \quad (62)$$

Rearranging terms we get

$$x_{t+T-3} = \frac{1}{\alpha_2} x_{t+T-2} - \frac{\alpha_1}{\alpha_2} x^* - \frac{\alpha_3}{\alpha_2} x_{t+T-1}.$$

Since $\frac{1}{\alpha_2} - \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_2} = 1$, which is contradictory with the assumption that x_{t+T-3} is not a linear combination of x_{t+T-1} , x_{t+T-2} and x^* . In this way, we can show recursively that all the policies corresponding to GP model are on the same straight line. \square

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